Turbulent Mixing of Reactive Gases

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Nomenclature

f'	=	velocity parameter, u/u_e
h_i	=	static enthalpy of ith species
H	=	stagnation enthalpy
W	=	molecular weight of mixture
α_{ik}	=	number of atoms of k in species i
β	=	pressure-gradient parameter
ϵ	=	eddy coefficient of viscosity
μ	=	coefficient of viscosity
ν_i', ν_i''	=	stoichiometric coefficients
ν_i	=	$(\nu_i'' - \nu_i')$
τ	=	shear stress
Υ_i	=	mass fraction of ith species
θ_k	=	element mass fractions
χ	=	similarity variable

Subscripts

e	=	values in the upper gas stream
i	=	ith species
j	=	values in the lower gas stream
k	=	elements composing species
t	=	turbulent flow values

All flow quantities considered in this analysis are slowly varying temporal mean quantities in the usual boundary-layer sense.

RECENT interest in the diffusion or deflagration mode of supersonic combustion has prompted the need for theoretical studies of the turbulent mixing problem for high-speed reactive flows. The basic flow model describing the fluidmechanical aspects of the flow within a combustion chamber is that of the turbulent mixing of two reacting streams of different mean velocities, temperatures, and composition. Since the describing equations for nonequilibrium turbulent shear flows have not as yet been established, it is necessary to limit the chemical model to either the frozen or equilibrium flow case when using the boundary-layer equations to describe the turbulent motion within the mixing region.

In an attempt to avoid the usual assumptions regarding the physical behavior of the flow, Coles[1] and Crocco[2] have investigated the merits of a purely mathematical procedure of transforming the governing equations. The emphasis in both their approaches is placed on obtaining the proper correspondence of the pressure and inertia terms in both the physical and transformed planes.

Following a similar approach, a transformation of variables is introduced according to

$$\frac{\bar\psi}{\psi} = \sigma(x) \qquad \frac{\bar\rho}{\rho}\frac{\partial\bar y}{\partial y} = \eta(x) \qquad \frac{d\bar x}{dx} = \xi(x) \tag{1}$$

where the stream functions are defined as

$$\psi = \int_0^y \rho u dy' \qquad \bar\psi = \int_0^{\bar y} \bar\rho\bar u d\bar y' \tag{2}$$

It should be noted that the barred quantities in Eqs. (1) and (2) refer to the transformed plane. Accordingly, it has been shown in Refs. 1 and 2 that

$$\frac{\bar u}{u} = \frac{\sigma}{\eta} = \frac{\bar u_e}{u_e} \qquad \rho v = \frac{\psi}{\sigma}\frac{d\sigma}{dx} + \frac{\xi}{\sigma}\bar\rho\bar v - \frac{\bar\rho\bar u}{\sigma}\frac{\partial\bar y}{\partial x} \tag{3}$$

$$\frac{d\bar p}{d\bar x} = \frac{\bar\rho_e\,\sigma^2}{\rho_e\eta^2\xi}\left[\frac{dp}{dx} + \rho_e u_e^2\,\frac{d\ln(\eta/\sigma)}{dx}\right] \tag{4}$$

$$\frac{\partial\bar\tau}{\partial\bar y} = \frac{\bar\rho}{\rho}\frac{\sigma^2}{\eta^2\xi}\left\{\frac{\partial\tau}{\partial y} - \frac{\psi}{\sigma}\frac{d\sigma}{dx}\frac{\partial u}{\partial y} + \frac{dp}{dx}\left(\frac{\rho\bar\rho_e}{\bar\rho\rho_e} - 1\right) + \bar\rho\, u_e^2\left(\frac{\bar\rho_e}{\bar\rho} - \frac{u^2}{u_e^2}\right)\frac{d\ln(\eta/\sigma)}{\partial x}\right\} \tag{5}$$

In order to establish the correspondence between the heat flux in the physical plane and that in the transformed plane, it is necessary to specify the manner in which the total enthalpies of the two systems are related. In general, it can be assumed that

$$H/H_e = \lambda + (1-\lambda)(\bar H/\bar H_e) \tag{6}$$

where the constant λ is to be determined consistent with the boundary conditions. Consequently, if a similar treatment is used on the energy equation as was applied to the momentum equation to yield Eqs. (4) and (5), there results

$$\frac{\partial}{\partial\bar y}(\bar q + \bar u\bar\tau) = \frac{\bar H_e}{(1-\lambda)H_e}\frac{\bar\rho\sigma}{\rho\eta\xi}\left\{\frac{\partial}{\partial y}(\dot q + u\tau) - \frac{\psi}{\sigma}\frac{d\sigma}{dx}\frac{\partial H}{\partial y}\right\} \tag{7}$$

Since the turbulent shear in free turbulent flows is much larger than the laminar shear, the latter can be neglected as being small away from any solid boundary. As a result,

$$\tau = \rho\epsilon(\partial u/\partial y) \qquad \bar\tau = \bar\rho\bar\epsilon(\partial\bar u/\partial\bar y) \tag{8}$$

$$\bar\tau = (\bar\rho^2\bar\epsilon/\rho^2\epsilon)(\sigma/\eta^2)\tau \tag{9}$$

$$\frac{\partial\bar\tau}{\partial\bar y} = \frac{\sigma}{\eta^3}\frac{\bar\rho}{\rho}\left\{\frac{\bar\rho^2\bar\epsilon}{\rho^2\epsilon}\frac{\partial\tau}{\partial y} + \tau\frac{\partial}{\partial y}\left(\frac{\bar\rho^2\bar\epsilon}{\rho^2\epsilon}\right)\right\} \tag{10}$$

In addition, for $P_r = P_{rt} = L_e = L_{et} = 1$, it can be shown that, since $\epsilon \gg \mu$ and $\bar\epsilon \gg \bar\mu$,

$$\dot q + u\tau = \rho\epsilon(\partial H/\partial y) \qquad \bar{\dot q} + \bar u\bar\tau = \bar\rho\bar\epsilon(\partial\bar H/\partial\bar y) \tag{11}$$

Thus

$$(\dot q + \bar u\bar\tau) = \frac{\bar H_e}{(1-\lambda)H_e}\frac{\bar\rho^2\bar\epsilon}{\rho^2\eta\epsilon}(\dot q + u\tau) \tag{12}$$

$$\frac{\partial}{\partial\bar y}(\bar q + \bar u\bar\tau) = \frac{\bar H_e}{(1-\lambda)H_e\eta^2}\frac{\bar\rho}{\rho}\left\{\frac{\bar\rho^2\bar\epsilon}{\rho^2\epsilon}\frac{\partial(\dot q + u\tau)}{\partial y} + (\bar q + u\tau)\frac{\partial}{\partial y}\left(\frac{\bar\rho^2\bar\epsilon}{\rho^2\epsilon}\right)\right\} \tag{13}$$

At this point it must be required that Eqs. (5) and (7) be identical to Eqs. (10) and (13), respectively, if the desired transformation is to yield consistency and compatibility between the physical and the transformed plane. An investigation of these relationships indicates that this is certainly true if

$$d\sigma/dx = 0 \qquad (\partial/\partial y)(\bar\rho^2\bar\epsilon/\rho^2\epsilon) = 0 \tag{14}$$

Consequently, it follows that (using asterisks to denote reference values)

$$\rho^2\epsilon/\bar\rho^2\bar\epsilon = \xi/\sigma\eta = \rho_*^2\epsilon_*/\bar\rho_*^2\bar\epsilon_* \tag{15}$$

$$\left(1 - \frac{\rho\bar\rho_e}{\rho_e\bar\rho}\right)\frac{dp}{dx} = \rho u_e^2\left(\frac{\bar\rho_e}{\bar\rho} - \frac{u^2}{u_e^2}\right)\frac{d\ln(\eta/\sigma)}{dx} \tag{16}$$

By defining a new parameter

$$\kappa(x) = 1 - \frac{[d\ln(\eta/\sigma)]}{d\ln u_e} \tag{17}$$

it is possible to reduce Eq. (16) to the form

$$\kappa(\bar\rho_e/\bar\rho) = (\rho_e/\rho) - (1-\kappa)(u^2/u_e^2) \tag{18}$$

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which represents the correspondence between the densities of the two systems. In addition to these relationships, it will be specified that the species and element mass fractions be invariant under the transformation.

It should be noted that, because of the presence of a streamwise pressure gradient and of the boundary-layer assumptions inherent in the governing conservation equations, an additional requirement exists, i.e., in both planes

$$\rho_e u_e(du_e/dx) = \rho_j u_j(du_j/dx) \tag{19}$$

which must be imposed on the solution to validate the use of boundary-layer theory to the problem at hand. This, in itself, is not a stringent requirement. However, the requirement becomes more stringent when similar solutions are sought.

Similar Solutions

Before the transformation of variables can be completed, it is necessary to consider certain aspects of the method of solution desired. Consistent with the usual approach, it is required that

$$\begin{aligned} \bar{\psi} &= \Lambda(\bar{x})f(\chi) \qquad & \bar{u}_e &= \bar{u}_e(\bar{x})f'(\chi) \\ \bar{H} &= \bar{H}_e\phi(\chi) \qquad & \Upsilon_i &= \Upsilon_{ie}Z_i(\chi) \end{aligned} \tag{20}$$

where $\bar{u}_e$ and $\bar{u}_j$ must have the same dependence on $\bar{x}$. Hence

$$f_j' = \bar{u}_j/\bar{u}_e = u_j/u_e = \text{const} \tag{21}$$

and the pressure-gradient requirement becomes

$$\rho_e/\rho_j = f_j'^2 = \bar{\rho}_e/\bar{\rho}_j \tag{22}$$

In addition, it becomes evident that the parameter κ must be a constant if the density profiles are to be similar.

Since Eq. (18) is automatically satisfied at the boundaries of the mixing region, κ may be evaluated at some reference condition (say the axis of mixing) according to

$$\kappa = \frac{(\rho_e/\rho_*) - f_*'^2}{(\bar{\rho}_e/\bar{\rho}_*) - f_*'^2} = \text{const} \tag{23}$$

Consistent with this result, it is possible to integrate Eq. (17) directly to yield

$$\eta/\sigma = (u_e/H_e^{1/2})^{1-k} \tag{24}$$

which, in turn, leads to

$$\xi/\sigma^2 = (\rho_*{}^2\epsilon_*/\bar{\rho}_*{}^2\bar{\epsilon}_*)(u_e/H_e^{1/2})^{1-k} \tag{25}$$

when substituted into Eq. (15).

Apart from the additional requirement imposed on the solution, consideration of a streamwise pressure gradient necessitates the choice of the low-speed fluid properties in determining the transformation. As in all pressure-gradient boundary-layer problems, it is necessary to specify the coupling existing between the density, stagnation enthalpy, and velocity. Along with Crocco,[2] it will be assumed that the properties of the low-speed fluid are such that

$$\bar{\rho}_e/\bar{\rho} \approx \bar{h}/\bar{h}_e \approx \bar{H}/\bar{H}_e = \phi(\chi) \tag{26}$$

Consequently, from Eqs. (6, 22, and 26), there results

$$\lambda = \frac{(H_j/H_e) - f_j'^2}{1 - f_j'^2} \tag{27}$$

By imposing that $\bar{u}_e = c_1\bar{x}^m$ and $\bar{\rho}_*{}^2\bar{\epsilon}_* = c_2\bar{x}^n$, the similarity analysis yields[3]

$$\Lambda = \left(\frac{\beta}{m}\,\bar{u}_e\,\bar{\rho}_*{}^2\bar{\epsilon}_*\bar{x}\right)^{1/2} \qquad \beta = \frac{2m}{1+n+m} \qquad \frac{\partial\chi}{\partial\bar{y}} = \frac{\bar{\rho}\bar{u}_e}{\Lambda} \tag{28}$$

which reduce the governing system of equations to†

$$f''' + ff'' = \beta(f'^2 - \phi) \tag{29}$$

$$\phi'' + f\phi' = 0 \tag{30}$$

$$z_i = [1/(1 - f_j'^2)][(z_{ij} - f_j'^2) + (1 - z_{ij})\phi] \tag{31}$$

subject to conditions $f' \to 1$, $\phi \to 1$, $z_i \to 1$, $f'' \to 0$, $\phi' \to 0$, and $z_i' \to 0$ for $\chi \to +\infty$, and $f' \to f_j'$, $\phi \to \phi_j$, $z_i \to z_{ij}$, $f'' \to 0$, $\phi' \to 0$, and $z_i' \to 0$ as $\chi \to -\infty$. It is of interest to note that Eqs. (29–31), together with the respective boundary conditions cited, are exactly the system solved for in Ref. 4. A complete tabulation of solutions to this system of equations can be found in Ref. 5.

At this point in the analysis, it is necessary to specify a realistic low-speed model for the turbulent exchange coefficient. As discussed in Ref. 3, it will suffice for the present analysis to adopt a model of the form $\bar{\rho}\bar{\epsilon} = k\bar{\rho}_e\bar{u}_e\bar{x}$, where the constant k is to be determined experimentally. Accordingly,

$$\bar{\rho}^2\bar{\epsilon} = \bar{\rho}_*{}^2\bar{\epsilon}_* = k(\bar{\rho}_e{}^2/\phi_*)\bar{u}_e\bar{x} \tag{32}$$

which results in

$$\Lambda = (\beta k/m\phi_*)^{1/2}\bar{\rho}_e\bar{u}_e\bar{x} \tag{33}$$

$$\partial\chi/\partial\bar{y} = (m\phi_*/\beta k)^{1/2}(\bar{\rho}/\bar{\rho}_e\bar{x}) \tag{34}$$

Inverse Transformation to Physical Plane

The results of the previous section have established the transformation of variables and the form of the transformed stream function and similarity parameter necessary for similarity, i.e.,‡

$$\left.\begin{aligned} \bar{x} &= \left[\frac{2\sigma^2\phi_*}{kH_e^{1/2}}\int_{x_1}^{x}\frac{\rho_*{}^2\epsilon_*}{\bar{\rho}^2}\left(\frac{u_e}{H_e^{1/2}}\right)^{1-2\kappa}dx'\right]^{1/2} \\ \bar{y} &= \sigma\left(\frac{u_e}{H_e^{1/2}}\right)^{1-\kappa}\int_0^y\frac{\rho}{\bar{\rho}}\,dy' \end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned} \bar{\psi} &= \left(\frac{\beta k}{m\phi_*}\right)^{1/2}\bar{\rho}_e\bar{u}_e\bar{x}f \\ \chi &= \frac{\sigma}{\bar{x}}\left(\frac{m\phi_*}{\beta k}\right)^{1/2}\frac{\rho_e}{\bar{\rho}_e}\left(\frac{u_e}{H_e^{1/2}}\right)^{1-\kappa}\int_0^y\frac{\rho}{\rho_e}\,dy' \end{aligned}\right\} \tag{36}$$

There remains now the task of formulating an inverse procedure whereby the results of the similarity analysis can be transformed back to the physical plane. Before this can be accomplished, however, it is necessary to evaluate κ and λ according to Eqs. (23) and (27).

From the velocity-enthalpy relation

$$\frac{h}{h_e} = \frac{H_e}{h_e}\left[\lambda + (1-\lambda)\phi - \frac{u_e^2}{2H_e}f'^2\right] = \frac{H_e}{h_e}\sum_i \Upsilon_{ie}z_i\frac{h_{ie}}{H_e}\frac{h_i}{h_{ie}} \tag{37}$$

it is evident that additional requirements are necessary for similarity, i.e., $u_e^2/2H_e = \text{const} \to 1$ and $h_{ie}/H_e = \text{const}$, since h_i/h_{ie} is considered to be only a function of χ. A convenient analytical representation for $h_i = h_i(T)$, as discussed in Ref. 4, is $h_i = C_iT^a$, where C_i and a are constants evaluated to best represent the actual $h_i = h_i(T)$ function in the temperature range arising in the mixing region. As a result,

† It has been assumed here that the Prandtl and Lewis numbers are equal to unity and that the element and species mass fractions external to the mixing region are constant. For greater detail, the reader is referred to Ref. 3.

‡ The transformation function σ has been retained in these expressions for the purpose of generality. For the low-speed transformed flow considered, it is possible to show that, within the approximation $(\bar{u}_e^2/2\bar{H}_e) \ll 1$, $\bar{\rho}_e$ becomes a constant and $n = 1 + m$.

it follows that $h_i/h_{ie} = (T/T_e)^a$, which yields

$$\frac{T}{T_e} = \left[\frac{\lambda + (1-\lambda)\phi - (u_e^2/2H_e)f'^2}{\sum_i \Upsilon_{ie} z_i \dfrac{h_{ie}}{H_e}}\right]^{1/a} \tag{38}$$

$$\frac{\rho_e}{\rho} = \frac{T}{T_e}\left(\frac{\sum_i \dfrac{\Upsilon_{ie} z_i}{W_i}}{\sum_i \dfrac{\Upsilon_{ie}}{W_i}}\right) \tag{39}$$

Equation (39), when evaluated at the reference condition, yields the density ratio required to determine κ from Eq. (23). Furthermore, since $(\rho_e/\rho_j) = f_j'^2$, the stagnation enthalpy ratio at the inner boundary ($\chi \rightarrow -\infty$) cannot be chosen arbitrarily but must satisfy the relation

$$g_j = \frac{H_j}{H_e} = \left(f_j'^2 \frac{W_j}{W_e}\right)^a \sum_i \Upsilon_{ie} z_{ij} \frac{h_{ie}}{H_e} + \frac{u_e^2}{2H_e} f_j'^2 \tag{40}$$

For the frozen-flow case, the species mass fractions follow directly from Eq. (31) once $\phi(\chi)$ has been determined from the solution of Eqs. (29) and (30). The other flow property ratios of interest, namely h/he, T/Te, and ρ_e/ρ, are evaluated from Eqs. (37–39), respectively. The normal coordinate in the physical plane is related to χ by

$$y = \left(\frac{\beta k}{m\phi_*}\right)^{1/2} \frac{\bar{\rho}_e}{\rho_e} \left(\frac{u_e}{He^{1/2}}\right)^{\kappa - 1} \frac{\bar{x}}{\sigma} \times \left[\kappa \int_0^\chi \phi d\chi' + (1-\kappa)\int_0^\chi f'^2 d\chi'\right] \tag{41}$$

Hence the transformation to the physical plane is complete once $\bar{x}$ is determined from Eq. (35).

For the equilibrium flow case, Eq. (31) can be used to determine the element mass fractions (see Ref. 3 for details) since, for this case, $z_i = z_k = \theta_k/\theta_{ke}$, where the element mass fractions θ_k are related to the species mass fractions by

$$\theta_k = \sum_{i=1}^{s} \alpha_{ik} \frac{W_k}{W_i} \Upsilon_i \tag{42}$$

Consequently, it is necessary to evaluate the species mass fractions and the temperature from the simultaneous solution of Eqs. (38) and (42), together with the relation for the equilibrium constants

$$K_{pl} = (pW)^{\sum_i \nu_{il}} \prod_{i=1}^{s} \left(\frac{\Upsilon_i}{W_i}\right)^{\nu_{il}} \tag{43}$$

In Eq. (43), the l subscript refers to the independent chemical reactions considered. The remainder of the inverse transformation follows a similar procedure as for the frozen-flow case once the species mass fractions and temperature have been determined.

It should be pointed out that, for temperatures of interest in supersonic combustion, it is possible to approximate the equilibrium behavior within the mixing region by adopting a flame-sheet combustion mode, as discussed in Ref. 4. This combustion model, which is based on the application of the low-speed diffusion flame concepts to high-speed flows, assumes that the reaction zone is of zero thickness and that the reacting species burn instantaneously upon reaching the sheet. These simplifications are justified provided the reaction rates are very large compared to the diffusion velocities of the species toward the sheet. The formulation and results of this approach can be found in Ref. 4.

References

[1] Coles, D. E., "The turbulent boundary layer in a compressible fluid," Rand Corp. Rept. R-403-PR (1962).

[2] Crocco, L., "Transformation of the compressible turbulent boundary layer with heat exchange," AIAA J. 1, 2723–2731 (1963).

[3] Casaccio, A., "Similar solutions for the turbulent mixing of reactive flows," Republic Aviation Corp. Rept. RAC-1860 (October 1964).

[4] Casaccio, A., "Similar solutions for the laminar mixing of reactive gases," AIAA J. 2, 1403–1409 (1964).

[5] Casaccio, A., "Similar solutions for the turbulent compressible mixing of two streams with streamwise pressure gradient," Republic Aviation Corp. Rept. RAC-882A (March 1963).

Deflections and Bending Moments of Rectangular Sandwich Panels with Clamped Edges under Combined Biaxial Compressions and Pressure

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Nomenclature

a, b, c	= length of panel; width of panel; core height
t_f	= face thickness
D_0	= flexural rigidity of panel
D_f	= flexural rigidity of one face sheet
G_c'	= shear stiffness of core
i, m, n	= odd integers unless noted
M_x, M_y	= sectional moments perpendicular to x axis and y axis, respectively
M_{xy}	= torsional moment per unit length of section perpendicular to x axis and y axes
N_x, N_y	= sectional loads perpendicular to x axis and y axes, respectively
p	= lateral pressure
W_{b1}	= bending deflection due to edge moment M_x
W_{b2}	= bending deflection due to edge moment M_y
$W_{b_{ss}}$	= bending deflection for simply supported edges
$W_{s_{ss}}$	= shear deflection for simply supported edges
$W_{T_{ss}} = W_{b_{ss}} + W_{s_{ss}}$	= total deflection for simply supported edges
x, y	= coordinates parallel to the length and width of the panel with origin at the center
X_n, Y_m	= function of x only, function of y only
∇^2	$= \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$
∇^4	$= \nabla^2 \nabla^2$
ϕ_m	$= m\pi/a$
β_n	$= \dfrac{n\pi}{b}$
μ	= Poisson's ratio

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